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Internal waves generated by a translating oscillating body

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The internal-wave system is calculated for a body oscillating transversely, and translating uniformly, through an infinite stratified fluid of constant Brunt–Väisälä frequency. A linearized, time-dependent analysis is used, in which the vertical displacement of a fluid element is the basic dependent variable. Axisymmetric slender-body theory for a homogeneous fluid is used to determine the time-dependent source and dipole distributions required to represent the motion of the body for excitation of the internal waves. The equations are solved by Fourier-transform techniques; and the internal-wave amplitude is evaluated in the far field by the method of stationary phase. The surfaces of constant phase are found to change character as the ratio of the oscillation frequency ω of the body to the Brunt–Väisälä frequency N varies through unity. Along preferred directions, the amplitude of the internal waves is found to decay inversely with distance to the $\frac{5}{6}$ power, whereas, for uniform translation, the amplitude of the internal waves falls off inversely with distance from the body. An asymptotic expression for the amplitude in preferred directions is calculated for several values of the ratio ω/N.

1. Introduction

As a body travels horizontally through a non-uniform stratified fluid, it may undergo pitch and heave oscillations that are superimposed on the otherwise uniform horizontal motion. The pitch and heave oscillations of the body relative to the ambient stratified fluid will induce internal waves. In this paper the amplitude of the internal waves excited by such a body both translating and undergoing heave oscillations through a quiescent, stratified fluid is calculated, and compared with the internal-wave amplitude excited by the uniform horizontal motion of the body.

In §2 a review is given of the literature on internal waves excited by an oscillating or a translating and oscillating body. In §3 a general formulation is presented for determining the internal-wave field excited in a stratified fluid by a slender translating and heaving body. For the case in which the ambient stratification is exponential, so that the Brunt–Väisälä frequency is constant, the solution is determined in integral form by Fourier-transform techniques. In §4 the asymptotic behaviour of this integral far from the source is examined.

In § 5 the asymptotic expressions for the internal waves excited by heave motion of a body are calculated, and the behaviour of these waves with changes in velocity of the body, and changes in the ratio of the heave frequency to the Brunt–Väisälä frequency is discussed.

2. Review of previous work on internal waves excited by an oscillating body

In a fluid with its density stably stratified in the vertical direction z, the natural oscillation frequency is the Brunt–Väisälä frequency, defined by

$$N = \left[\frac{g}{\rho_0}\left|\frac{d\rho_0}{dz}\right|\right]^{\frac{1}{2}}. \tag{1}$$

g is the acceleration of gravity, and $d\rho_0/dz$ is the local density gradient. The time scale $T_0 = 2\pi/N$ is the local characteristic time for internal waves to occur naturally in the fluid.

Mowbray & Rarity (1967) examined experimentally and theoretically the two-dimensional internal-wave system excited by a cylinder oscillating in a stratified fluid where the Brunt–Väisälä frequency is approximately constant.† They found that internal waves can be excited provided the frequency of oscillation ω of the body does not exceed the Brunt–Väisälä frequency. For frequencies greater than the Brunt–Väisälä frequency, internal-wave oscillations will not be sustained. They also found that, when internal waves are excited, significant amplitudes in the internal-wave system occur only in regions emanating along preferred directions from the oscillating cylinder. The preferred directions are determined by the angle θ, measured from horizontal, where

$$\sin\theta = \pm\,\omega/N. \tag{2}$$

The variation of the angle θ with variation in the frequency ω of the oscillating body was measured, and it was verified experimentally that the Brunt–Väisälä frequency N is the high-frequency cut-off.

In a theoretical paper, Hendershott (1969) determined the amplitude of the internal waves at large distances from a sphere which excites these waves by 'breathing' fluid in and out at a given frequency $\omega < N$. His analysis included the effects of rotation of the fluid, as well as constant-N stratification. At long times after the sphere has begun to pulsate, if the effects of rotation are neglected, an axisymmetric flow field develops, very similar to that described by Mowbray & Rarity. Internal waves of significant amplitude are almost entirely confined to regions determined by the preferred directions. In this case, the regions are the axisymmetric ones defined by vertical cones of constant angle θ, tangential to the sphere above and below. Along the preferred direction, the magnitude of the vertical velocity decreases inversely as the square root of the radial distance from the sphere. In regions outside the preferred ones, the amplitude approaches zero inversely as the square of the radial distance from the sphere.

† A constant Brunt–Väisälä frequency N implies an exponentially stratified fluid with density $\rho_0(z) = \rho_{00}\exp(-N^2z/g)$, where ρ_{00} is the density at $z = 0$, and z is measured positive upward.

Stevenson & Thomas (1969) extended the experimental studies of Mowbray & Rarity to the case of an oscillating cylinder moving with a uniform velocity in an arbitrary direction. They derived the dispersion relation for waves generated by the cylinder and tested some of the predictions of this relation experimentally.

Stevenson (1973) further extended these studies to the case of transient motions and curved motion of a body in a stratified fluid. He found reasonable agreement between theoretical predictions and experimental observations of the phase configuration of the internal waves, but did not determine the amplitude of these waves.

Analysis of the internal waves produced by a uniformly moving dipole, or by a quadrupole source in a stratified fluid, was performed by Miles (1971). Such sources are usually considered to produce the dominant terms in the description of the internal-wave disturbances excited by a body and by the collapse of the turbulent wake behind a body. Miles solved the linearized equations for the amplitude of internal waves excited by a dipole oriented horizontally and translating uniformly along its axis. He analysed both the constant-N case and the case of a thin thermocline, using Fourier-transform analysis and stationary-phase asymptotic evaluation of the resulting integral expressions for the solution. In this paper, the analysis of Miles is extended by using slender-body theory to formulate explicitly the boundary conditions for a body translating uniformly in the horizontal direction and oscillating with a heaving motion. The dipole solution which Miles determined in the constant-N case is the same as the solution determined in this paper for a small body in uniform translation.

Keller & Munk (1970) also examined the internal-wave amplitudes generated by a uniformly translating excitation source in a stratified medium of variable Brunt–Väisälä frequency. Their analysis was rather general, applicable to wave systems generated by a moving source in any dispersive, anisotropic medium. But it did not explicitly relate the excitation source to the body dynamics, as we do.

McLaren *et al.* (1973) reported measurements of the amplitude of the internal waves excited by an oscillating body. They found that the amplitude along the preferred directions appears to fall off with increasing distance as $(a/R)^{\frac{1}{2}}$, where R is the radial distance from the bobbing sphere, and a is the radius of the sphere.

Rao (1973), in a theoretical paper, examined the wave pattern generated by a two-dimensional translating and oscillating forcing effect in a rotating fluid. He used the methods developed by Lighthill (1959, 1967) for waves in a dispersive and anisotropic fluid. He also noted that the analogy which exists between the dynamics of rotating fluids and stratified fluids may be used to relate some of his results to similar phenomena for two-dimensional effects in stratified fluids.

3. Formulation of the solution for a translating, heaving body

The model used to make an estimate of internal waves from heave oscillations is one in which the assumptions are generally similar to those used in the analyses

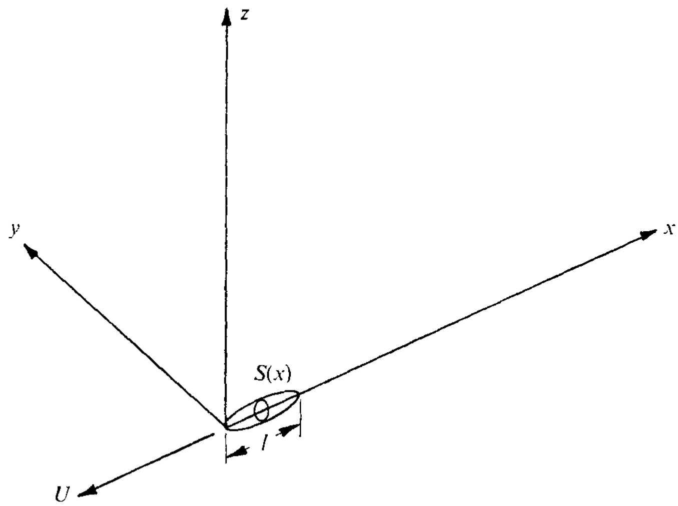

FIGURE 1. Reference axes.

discussed in §2. The stratified fluid is assumed to have an exponentially increasing density with increasing depth so that the Brunt–Väisälä frequency is constant, the so-called constant-N model. The Boussinesq approximation is made: in this approximation, density variations are considered only in the buoyancy terms in the equations of motion, the density being taken as constant in the inertial terms. A small-amplitude or linearized analysis is used for the calculation of internal waves, and slender-body theory is used to apply boundary conditions at the body. The body is assumed to be translating uniformly in the horizontal direction (chosen to be the negative x direction) and to be undergoing small-amplitude heaving oscillations. The z axis is chosen positive upward, with the y axis horizontal and transverse to the translation (see figure 1).

The linearized equation for the vertical displacement ζ of a fluid element, due to internal waves excited by a source distribution of magnitude Q, is (see e.g. Miles 1971)

$$[\partial_z^2\partial_t^2+(\partial_t^2+N^2)(\partial_x^2+\partial_y^2)]\,\zeta = \partial_z\partial_t Q. \tag{3}$$

t is time, and the symbol ∂ with a subscript denotes differentiation with respect to the variable used in the subscript.

Slender-body theory can be used to determine the internal-wave field for an axisymmetric body of arbitrary cross-sectional area $S(x)$. When the transverse dimension a of a body is much less than the length scale $1/k = U/N$ (based upon the velocity U of the body and the Brunt–Väisälä frequency N, so that $ka \ll 1$), the fluid is homogeneous locally around the body, and the flow is potential. There will be a region for which $a \ll R \ll 1/k$, where R is the distance from the body, in which the flow is unaffected by the stratification (as described by Miles 1971). In this region, in a linearized analysis, the flow-field resulting from arbitrary motion of an axisymmetric body along the x axis can be represented by a general source–sink distribution along the x axis. This source distribution is determined from the boundary condition that the fluid velocity be zero in a direction normal to the body surface, and can be determined using slender-body theory (see e.g. Liepmann & Roshko 1957, ch. 9; Frankl & Karpovich 1953, ch. 2).

When the body is both translating and heaving, in a linearized analysis, the flow field relative to the axisymmetric body can be divided into two superposable components: the axial and the cross flow. For a uniformly translating body of arbitrary cross-section $S(x)$ for $0 \leqslant x \leqslant l$, the source distribution that yields the potential flow over the body is

$$Q = \int_0^l U(dS/d\xi)\, d\xi$$

multiplied by a point source at $x+Ut = \xi$, $y = 0$, $z = 0$. If we denote by $\zeta^{(a)}$ the solution of (3) for this source distribution, then $\zeta^{(a)}$ satisfies

$$[\partial_z^2 \partial_t^2 + (\partial_t^2 + N^2)(\partial_x^2 + \partial_y^2)]\, \zeta^{(a)} = \partial_z \partial_t \left[U \int_0^l d\xi \frac{dS(\xi)}{d\xi} \delta(\xi - x - Ut)\, \delta(y)\, \delta(z) \right]. \quad (4)$$

$\zeta^{(a)}$ will be called the axial-flow solution.

In a potential flow, when the cross-flow boundary condition is applied for a body heaving with amplitude h and frequency ω, the cross-flow component can be represented by a distribution of vertically oriented dipoles (see Frankl & Karpovich 1953). If the same distribution of dipoles is used in the stratified case, the equation satisfied by the cross-flow internal-wave amplitude $\zeta^{(c)}$ is

$$[\partial_z^2 \partial_t^2 + (\partial_t^2 + N^2)(\partial_x^2 + \partial_y^2)]\, \zeta^{(c)}$$
$$= \partial_z^2 \partial_t \left[2h\omega \exp(i\omega t) \int_0^l d\xi\, S(\xi)\, \delta(\xi - x - Ut)\, \delta(y)\, \delta(z) \right]. \quad (5)$$

Let $G^{(a)}(x, y, z, t;\ \xi)$ be the Green's function for the vertical displacement of a fluid particle in an internal wave excited by a point source at position ξ along the x axis and translating with velocity $-U$ along this axis. Then

$$\zeta^{(a)}(x,y,z,t) = \int_0^l d\xi\, U \frac{dS(\xi)}{d\xi} G^{(a)}(x,y,z,t;\xi). \quad (6)$$

$G^{(a)}$ satisfies

$$[\partial_z^2 \partial_t^2 + (\partial_t^2 + N^2)(\partial_x^2 + \partial_y^2)]\, G^{(a)} = \partial_z \partial_t [\delta(\xi - x - Ut)\, \delta(y)\, \delta(z)]. \quad (7)$$

Similarly, let $G^{(c)}(x, y, z, t;\ \xi)$ represent the Green's function for the vertical displacement of a fluid particle produced by a vertically oriented dipole oscillating in strength with frequency ω and translating with velocity $-U$ along the x axis. Then

$$\zeta^{(c)}(x,y,z,t) = \int_0^l d\xi\, 2h\omega S(\xi)\, G^{(c)}(x,y,z,t;\xi). \quad (8)$$

$G^{(c)}$ satisfies

$$[\partial_z^2 \partial_t^2 + (\partial_t^2 + N^2)(\partial_x^2 + \partial_y^2)]\, G^{(c)} = \partial_z^2 \partial_t [\exp(i\omega t)\, \delta(\xi - x - Ut)\, \delta(y)\, \delta(z)]. \quad (9)$$

The simplest case to treat, and a case which will display the important features of the internal-wave system excited by a translating and heaving body, is when the Brunt–Väisälä frequency N is constant. Lighthill (1959, 1967) presented general methods for solution by Fourier-transform techniques of problems of the type given by (9). Use of the Fourier integral, where $\hat{x} = x-\xi$,

$$G^{(c)}(x,y,z,t) = \frac{1}{(2\pi)^3}\exp(i\omega t)\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma\, \tilde{G}^{(c)}(\alpha,\beta,\gamma,t) \times \exp[i(\alpha\hat{x}+\beta y+\gamma z)] \quad (10)$$

transforms (9), when N is constant, into

$$\{(\alpha U+\omega)^2\gamma^2+[(\alpha U+\omega)^2-N^2](\alpha^2+\beta^2)\}\tilde{G}^{(c)} = -i\gamma^2(\alpha U+\omega)\exp(i\alpha Ut). \quad (11)$$

Equation (11) has the formal solution

$$G^{(c)} = \frac{\exp(i\omega t)}{(2\pi)^3}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} d\gamma \frac{[-i\gamma^2(\alpha U+\omega)]\exp[i(\alpha\hat{x}+\beta y+\gamma z)+i\alpha Ut]}{(\alpha U+\omega)^2\gamma^2+[(\alpha U+\omega)^2-N^2](\alpha^2+\beta^2)}. \quad (12)$$

Equations (6) and (8), respectively, give the internal-wave amplitudes generated by uniform translation and by transverse heave oscillations, of a slender body. For a slender body, the ratio of the transverse dimension a of the body to its length l is small. If, in addition, the body is small, $lN/U \ll 1$. For a small, slender body, Miles (1971) pointed out that the internal-wave amplitude excited by translation is proportional to the Green's function for a uniformly translating, horizontally oriented dipole. In our notation this Green's function is $\partial_x G^{(a)}$, with $\xi = 0$, and $\zeta^{(a)} \simeq Ul\bar{S}\,\partial_x G^{(a)}$, where $\bar{S}$ is the mean cross-sectional area of the body.

Similarly, for a small, slender body, the internal-wave amplitude due to transverse oscillations will be proportional to the Green's function $G^{(c)}$, with $\xi = 0$, $\zeta^{(c)} \simeq 2h\omega l\bar{S}G^{(c)}$. Because of this proportionality for a small, slender body, the properties of the Green's function will be ascribed to the internal-wave amplitude $\zeta^{(c)}$.

The amplitude of internal waves due to transverse oscillations of a translating and oscillating slender body, compared with that of those due to only a translating body, is given by the ratio

$$\frac{\zeta^{(c)}}{\zeta^{(a)}} = \frac{2h\omega}{U}\frac{\int_0^l S(\xi)\,G^{(c)}(x,y,z,t;\xi)\,d\xi}{\int_0^l S(\xi)\,\partial_x G^{(a)}(x,y,z,t;\xi)\,d\xi}. \quad (13)$$

To estimate this quantity, one requires values for the ratio $h\omega/U$. In addition, the asymptotic values of the two integrals above are required for large values of $R = [(\hat{x}+Ut)^2+y^2+z^2]^{\frac{1}{2}}$. Miles (1971) evaluated $\partial_x G^{(a)}$ asymptotically. The asymptotic evaluation of $G^{(c)}$ is discussed in §4.

4. Asymptotic evaluation of the Green's function for a translating, oscillating dipole

The integral over γ in (12) is evaluated directly by using residue theory, as described by Lighthill. The condition that the disturbance source excites only outgoing waves at large distances from the disturbance is required to make the integrals unambiguous. When this is done, the integral for $G^{(c)}$ becomes

$$G^{(c)} = \frac{\exp(i\omega t)}{2(2\pi)^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \,\frac{(\beta^2+\alpha^2)^{\frac{1}{2}}\left[1-\left(\frac{\alpha U}{N}+\frac{\omega}{N}\right)^2\right]^{\frac{1}{2}}}{N(\alpha U/N+\omega/N)^2}$$
$$\times \exp\left\{i\left[\alpha(\hat{x}+Ut)+\beta y+(\alpha^2+\beta^2)^{\frac{1}{2}}\frac{[1-(\alpha U/N+\omega/N)^2]^{\frac{1}{2}}}{(\alpha U/N+\omega/N)}|z|\right]\right\}. \quad (14)$$

This integral can be rewritten, if we define a phase function ϕ and an amplitude function f, as

$$\phi\left(\alpha,\beta;\frac{\bar{x}}{R},\frac{y}{R},\frac{z}{R};\frac{U}{N},\frac{\omega}{N}\right) = \alpha\frac{\bar{x}}{R}+\beta\frac{y}{R}+(\alpha^2+\beta^2)^{\frac{1}{2}}\frac{\left[1-\left(\alpha\frac{U}{N}+\frac{\omega}{N}\right)^2\right]^{\frac{1}{2}}}{(\alpha U/N+\omega/N)}\frac{|z|}{R}, \quad (15)$$

$$f\left(\alpha,\beta;\frac{U}{N},\frac{\omega}{N}\right) = \frac{(\alpha^2+\beta^2)^{\frac{1}{2}}[1-(\alpha U/N+\omega/N)^2]^{\frac{1}{2}}}{N(\alpha U/N+\omega/N)^2}, \quad (16)$$

where $\bar{x} = \hat{x}+Ut$ and $R = [\bar{x}^2+y^2+z^2]^{\frac{1}{2}}$. Then $G^{(c)}$ becomes

$$G^{(c)} = \frac{\exp(i\omega t)}{2(2\pi)^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta f\left(\alpha,\beta;\frac{U}{N},\frac{\omega}{N}\right)\exp\left[iR\phi\left(\alpha,\beta;\frac{\bar{x}}{R},\frac{y}{R},\frac{|z|}{R};\frac{U}{N},\frac{\omega}{N}\right)\right], \quad (17)$$

4.1. *Points of stationary phase*

In this form the method of stationary phase can be applied to evaluate the integral in (17). The method requires that the amplitude function be smooth, and slowly varying compared with the oscillatory exponential factor. Although f is singular at $\alpha = -\omega/U$, the singularity does not contribute to the integral in (17). The method also requires R to be large, so that $RN/U \gg 1$. Only near points where the phase is stationary will the contribution to the integral be large; these points are given by the conditions

$$\left.\begin{aligned}\frac{\partial\phi}{\partial\alpha} &= \frac{\bar{x}}{R}+\frac{|z|}{R}\frac{\alpha\omega/N-\alpha(\alpha U/N+\omega/N)^3-\beta^2U/N}{(\alpha U/N+\omega/N)^2(\alpha^2+\beta^2)^{\frac{1}{2}}[1-(\alpha U/N+\omega|N)^2]^{\frac{1}{2}}} = 0,\\ \frac{\partial\phi}{\partial\beta} &= \frac{y}{R}+\frac{|z|}{R}\frac{\beta}{(\alpha^2+\beta^2)^{\frac{1}{2}}}\frac{[1-(\alpha U/N+\omega/N)^2]^{\frac{1}{2}}}{(\alpha U/N+\omega/N)} = 0.\end{aligned}\right\} \quad (18)$$

Solution of the second of these relations gives β as a function of α for points of stationary phase:

$$\beta_0 = -|\alpha|\frac{(\alpha U/N+\omega/N)\,y/|z|}{[1-(\alpha U/N+\omega/N)^2(y^2/z^2+1)]^{\frac{1}{2}}}. \quad (19)$$

Substitution of β_0 from (19) into the first of (18) gives the relation for determining the values of α at which the phase is stationary:

$$\frac{\partial\phi}{\partial\alpha}(\alpha,\beta_0)=\frac{\bar{x}}{R}+\frac{|z|}{R}\frac{\operatorname{sgn}(\alpha)\,[\omega/N-(\alpha U/N+\omega/N)^3(y^2/z^2+1)]}{(\alpha U/N+\omega/N)^2[1-(\alpha U/N+\omega/N)^2(y^2/z^2+1)]^{\frac{1}{2}}}=0. \quad (20)$$

With $\xi\equiv(\alpha U/N+\omega/N)$ the square of this relation may be written as

$$\xi^6-\xi^4\frac{\bar{x}^2z^2}{r^2R^2}-\xi^3\frac{2(\omega/N)\,z^2}{R^2}+\left(\frac{\omega}{N}\right)^2\frac{z^4}{r^2R^2}=0, \quad (21)$$

where $r^2=y^2+z^2, R^2=r^2+\bar{x}^2, \bar{x}=x+Ut$. ($\xi$ is not related to the ξ used in §3.)

The values of ξ which satisfy (21) will depend upon the ratio ω/N, and upon the direction from the body, but are independent of U/N. However, only the real roots of (21) are points of stationary phase, and of these roots only those which satisfy (20) are stationary-phase points.

An important qualitative difference occurs between the internal-wave system excited by a translating, oscillating body and that excited by a body only translating. A body only translating will generate internal waves of significant amplitude solely in the region behind the body. The estimate of the internal waves for a uniformly translating dipole given by Miles is for those waves downstream of the body. As Miles pointed out, "internal gravity waves . . . appear only downstream of their sources in a steady flow". By contrast, a body both translating and oscillating can excite internal waves both upstream and downstream of the body when $\omega/N<1$.

This qualitative difference can be seen by examination of (20). When $\omega/N=0$, this relation determines the points of stationary phase for a body only translating. Equation (21) reduces to a quadratic equation, yielding two possible points of stationary phase. When $\bar{x}<0$ (i.e. upstream), neither of these roots satisfies (18); and there is no contribution to integral (14) by the method of stationary phase. But, when $\bar{x}>0$ (i.e. downstream), both roots satisfy (20), and there are two contributions. Hence, the internal waves of significant amplitude, as determined by this method, occur only downstream of the translating body.

When $0<\omega/N<1$, roots of (21) satisfy (20) for $\bar{x}>0$ or $\bar{x}<0$, depending on whether $\operatorname{sgn}(\alpha)\,[\omega/N-(\alpha U/N+\omega/N)^3\,(y^2/z^2+1)]$ is <0 or >0. Hence, there are contributions to integral (14) by the method of stationary phase for both $\bar{x}>0$ and $\bar{x}<0$; and internal waves propagate both downstream and upstream of the translating, oscillating body.

For arbitrary ω/N and general directions, it is not possible to obtain the roots of (21) in analytical form. Therefore, a numerical program was written to calculate these roots. For this purpose, (21) was rewritten, using the directions θ and Φ defined as

$$\left.\begin{aligned}\bar{x}&=R\cos\theta, & 0&<R<\infty,\\ y&=R\sin\theta\cos\Phi, & 0&\leqslant\theta\leqslant\pi,\\ z&=R\sin\theta\sin\Phi, & 0&\leqslant\Phi\leqslant 2\pi.\end{aligned}\right\} \quad (22)$$

$\Phi\backslash\theta$	$\frac{1}{8}\pi$	$\frac{1}{4}\pi$	$\frac{3}{8}\pi$	$\frac{1}{2}\pi$
$\frac{1}{8}\pi$	0·1211659 −0·1432749 0·3673576 −0·3307695	0·3295014 0·1690617	0·2853967 0·2073608	0·24465820 0·24465820
$\frac{1}{4}\pi$	0·1655123 −0·184878 0·6684691 −0·6335528	0·5708069 0·2354168	0·4594712 0·2978747	0·36840315 0·36840315
$\frac{3}{8}\pi$	0·1896718 −0·2084446 0·8691646 −0·8345031	0·7286919 0·2718658	0·5672829 0·3484745	0·42882689 0·42882689
$\frac{1}{2}\pi$	0·1974823 −0·2161161 0·9395992 −0·9049894	0·7837959 0·2836835	0·6041047 0·3649947	0·46415888 0·46415888

TABLE 1. Roots of (23) for $\omega/N = 0{\cdot}1$

$\Phi\backslash\theta$	$\frac{1}{8}\pi$	$\frac{1}{4}\pi$	$\frac{3}{8}\pi$	$\frac{1}{2}\pi$
$\frac{1}{8}\pi$	0·3787832 0·2849855	0·3295014 0·1690617	No real roots	0·41835964 0·41835964
$\frac{1}{4}\pi$	0·7024413 0·3701491	0·6885052 0·500	0·6649701 0·5785319	0·62996053 0·62996053
$\frac{3}{8}\pi$	0·9093992 0·4211324	0·8709282 0·5734475	0·8173129 0·6751176	0·75289354 0·75289354
$\frac{1}{2}\pi$	0·4378267 −0·6479774 0·9814831 −0·7071068	0·9332624 0·5977362	0·8684060 0·7071068	0·79370053 0·79370053

TABLE 2. Roots of (23) for $\omega/N = 0{\cdot}5$

$\Phi\backslash\theta$	$\frac{1}{8}\pi$	$\frac{1}{4}\pi$	$\frac{3}{8}\pi$	$\frac{1}{2}\pi$
$\frac{1}{8}\pi$	No real roots	No real roots	No real roots	0·47890223 0·47890223
$\frac{1}{4}\pi$	0·7068760 0·4623917	0·7054760 0·6269878	No real roots	0·72112478 0·72112478
$\frac{3}{8}\pi$	0·9212846 0·5204584	0·9125978 0·7009841	0·8947992 0·8032004	0·86184794 0·86184794
$\frac{1}{2}\pi$	0·9951375 0·5400064	0·9799896 0·7284082	0·9525777 0·8393701	0·90856030 0·90856030

TABLE 3. Roots of (23) for $\omega/N = 0{\cdot}75$

$\Phi\backslash\theta$	$\frac{1}{8}\pi$	$\frac{1}{4}\pi$	$\frac{3}{8}\pi$	$\frac{1}{2}\pi$
$\frac{1}{8}\pi$	No real roots	No real roots	No real roots	0·50890956 0·50890956
$\frac{1}{4}\pi$	0·7016033 0·5173969	No real roots	No real roots	0·76630944 0·76630944
$\frac{3}{8}\pi$	0·9238272 0·5753723	0·9235923 0·7753912	0·9226031 0·8777110	0·91585011 0·91585011
$\frac{1}{2}\pi$	0·9991807 0·5958825	0·9960426 0·8019749	0·9876310 0·9120642	0·96548938 0·96548938

TABLE 4. Roots of (23) for $\omega/N = 0\cdot9$

Thus (21) becomes

$$\xi^6 - \xi^4 \cos^2\theta \sin^2\Phi - \xi^3\left(2\frac{\omega}{N}\right)\sin^2\theta\sin^2\Phi + \left(\frac{\omega}{N}\right)^2\sin^2\theta\sin^4\Phi = 0. \quad (23)$$

Tables 1–4 present, for various directions θ and Φ, values of the roots ξ_j to (23) for $\omega/N = 0\cdot1$, $0\cdot5$, $0\cdot75$, and $0\cdot9$, respectively.

4.2. *Asymptotic evaluation of the Green's function for arbitrary directions*

In the neighbourhood of a point of stationary phase (α_j, β_j), the phase function ϕ can be expanded to second order in a two-dimensional Taylor series as

$$\phi(\alpha,\beta) \simeq \phi(\alpha_j,\beta_j) + \tfrac{1}{2}[\phi_{\alpha\alpha}(\alpha_j,\beta_j)(\alpha-\alpha_j)^2 + 2\phi_{\alpha\beta}(\alpha_j,\beta_j)(\alpha-\alpha_j)(\beta-\beta_j) + \phi_{\beta\beta}(\alpha_j,\beta_j)(\beta-\beta_j)^2]. \quad (24)$$

Subscripts denote differentiation; and the explicit dependence of the phase function upon the directions in space and upon the parameters U/N and ω/N has been suppressed. By a simple rotation of axes in wavenumber space

$$\alpha-\alpha_j = \hat{\eta}\cos\psi - \hat{\zeta}\sin\psi, \quad \beta-\beta_j = \hat{\eta}\sin\psi + \hat{\zeta}\cos\psi, \quad (25)$$

the phase function can be written as

$$\phi(\alpha,\beta) \simeq \phi(\alpha_j,\beta_j) + \tfrac{1}{2}K_+(\alpha_j,\beta_j)\hat{\eta}^2 + \tfrac{1}{2}K_-(\alpha_j,\beta_j)\hat{\zeta}^2. \quad (26)$$

$K_\pm$ are the principal curvatures in wavenumber space, and the angle ψ has been selected to eliminate the 'cross term', which would be multiplied by $\hat{\zeta}\hat{\eta}$ in the general expression, i.e.

$$\tan 2\psi = \frac{2\phi_{\alpha\beta}(\alpha_j,\beta_j)}{\phi_{\alpha\alpha}(\alpha_j,\beta_j) - \phi_{\beta\beta}(\alpha_j,\beta_j)}. \quad (27)$$

The integral for $G^{(c)}$ in (14) can be approximated by the sum over all points of stationary phase of the contributions to the integral at each point. To obtain an explicit expression for $G^{(c)}$, various quantities must be evaluated at the stationary-phase point (α_j, β_j). Using ξ_j again to represent a root of (23), we find

$$\left.\begin{aligned}
\phi_{\alpha\alpha}(\alpha_j,\beta_j) &= \frac{|z|}{R}\frac{1}{|\alpha_j|}\frac{\xi_j\,(\xi_j^3 y^2/z^2 - 2\alpha_j U/N)(1-\xi_j^2 r^2/z^2) + (\alpha_j U/N)^2(2-3\xi_j^2)}{\xi_j^3(1-\xi_j^2)(1-\xi_j^2 r^2/z^2)^{\frac{1}{2}}} \\
&\equiv \frac{|z|}{R}\frac{1}{|\alpha_j|}\hat{\phi}_{\alpha\alpha}, \\
\phi_{\alpha\beta}(\alpha_j,\beta_j) &= \frac{y}{R}\frac{1}{|\alpha_j|}\frac{[-\omega/N + 2\xi_j - \xi_j^3 r^2/z^2]\,\xi_j^2\,[1-\xi^2 r^2/z^2]^{\frac{1}{2}}}{\xi_j^3(1-\xi_j^2)[1-\xi_j^2 r^2/z^2]^{\frac{1}{2}}} \equiv \frac{y}{R}\frac{1}{|\alpha_j|}\hat{\phi}_{\alpha\beta}, \\
\phi_{\beta\beta}(\alpha_j,\beta_j) &= \frac{|z|}{R}\frac{1}{|\alpha_j|}\frac{\xi_j^2[1-\xi_j^2 r^2/z^2]^2}{\xi_j^3(1-\xi_j^2)[1-\xi_j^2 r^2/z^2]^{\frac{1}{2}}} \equiv \frac{|z|}{R}\frac{1}{|\alpha_j|}\hat{\phi}_{\beta\beta}.
\end{aligned}\right\} \tag{28}$$

$\hat{\phi}_{\alpha\alpha}$, $\hat{\phi}_{\beta\beta}$ and $\hat{\phi}_{\alpha\beta}$ are defined by comparison with the relations above. Then

$$K_{\pm} = \frac{1}{2|\alpha_j|}\frac{|z|}{R}\left\{(\hat{\phi}_{\alpha\alpha}+\hat{\phi}_{\beta\beta}) \pm \left[(\hat{\phi}_{\alpha\alpha}-\hat{\phi}_{\beta\beta})^2 + 4\frac{y^2}{z^2}\hat{\phi}_{\alpha\beta}^2\right]^{\frac{1}{2}}\right\}. \tag{29}$$

Also,

$$\phi(\alpha_j,\beta_j) = \alpha_j\left\{\frac{x}{R} + \frac{[1-\xi_j^2(y^2/z^2+1)]^{\frac{1}{2}}}{\xi_j}\frac{|z|}{R}\right\}, \tag{30}$$

$$f(\alpha_j,\beta_j) = \frac{|\alpha_j|}{N}\frac{[1-\xi_j^2]}{\xi_j^2[1-\xi_j^2(y^2/z^2+1)]^{\frac{1}{2}}} \equiv \frac{|\alpha_j|}{N}\hat{f}. \tag{31}$$

Then expression (17) for $G^{(c)}$ becomes

$$G^{(c)} = \frac{\exp(i\omega t)}{2(2\pi)}\sum_j \frac{1}{R}\frac{\alpha_j^2}{N}\frac{R\hat{f}}{z}\frac{\exp\{\frac{1}{4}i\pi[\operatorname{sgn}(K_+)+\operatorname{sgn}(K_-)]\}}{\{\hat{\phi}_{\alpha\alpha}\hat{\phi}_{\beta\beta} - y^2/z^2\hat{\phi}_{\alpha\beta}^2\}^{\frac{1}{2}}} \times \exp\left[iR\alpha_j\left\{\frac{x}{R} + \frac{[1-\xi_j^2(y^2/z^2+1)]^{\frac{1}{2}}}{\xi_j}\frac{|z|}{R}\right\}\right]. \tag{32}$$

There are two important features to note from (32). First, for an arbitrary direction relative to the body, the fall-off of the amplitude with distance from the body is as $1/R$. The fall-off with distance of the internal-wave system from a uniformly translating body is also as $1/R$. Hence, from (13) the relative effect of the heave motions will be independent of radius and will be of the order of the ratio of the transverse velocity of the body to the translational motion of the body, which mostly likely is small.

Second, there will be special directions in space, analogous to the preferred directions for an oscillating body at rest, along which the asymptotic expression for $G^{(c)}$, given by (32), fails to be valid. In these directions, one of the principal curvatures in wavenumber space vanishes (say K_-), and the factor in the denominator of (32) becomes zero:

$$\hat{\phi}_{\alpha\alpha}\hat{\phi}_{\beta\beta} - y^2/z^2\,\hat{\phi}_{\alpha\beta}^2 = 0. \tag{33}$$

In these directions the procedure used to derive (32) breaks down. These special or preferred directions depend upon the ratio ω/N alone, and are independent of the ratio U/N.

4.3. *Asymptotic evaluation of the Green's function for preferred directions*

In directions determined by (33), the expansion of the phase function (24) must be altered to retain third-order terms. The procedure followed here is also outlined by Lighthill (1959). Again, we use the rotation of axes in wavenumber space (25), and choose the angle ψ to eliminate the cross term in the second-order expansion term (27). Then the phase-function expansion becomes

$$\phi(\alpha,\beta) \cong \phi(\alpha_j,\beta_j)+\tfrac{1}{2}K_+\hat{\eta}^2+\tfrac{1}{6}[\lambda_{\eta\eta\eta}\hat{\eta}^3+\lambda_{\eta\eta\zeta}\hat{\eta}^2\zeta+\lambda_{\eta\zeta\zeta}\hat{\eta}\zeta^2+\lambda_{\zeta\zeta\zeta}\zeta^3], \qquad (34)$$

where we have taken K_- to be the principal curvature that vanishes.

The important point is that the phase function now has both a quadratic and a cubic behaviour in the neighbourhood of the point of stationary phase. In a general direction in space, as shown in (26), the phase function increases quadratically with 'distance' in wavenumber space. The angle ψ represents the rotation required in wavenumber space to bring the axes into coincidence with the principal directions of the quadratic form for the phase function. In preferred directions in space, one of the values of the principal curvatures in wavenumber space vanishes. Therefore, in one of the principal directions, the $\hat{\eta}$ direction in this case, the phase function increases quadratically, whereas it increases cubically in the other direction. Lighthill asserted that only the cubic term in ζ needs to be retained, to obtain an approximate expression for the behaviour of the integral (17) in the preferred directions in space. When this is done,

$$\phi(\alpha,\beta) = \phi(\alpha_j,\beta_j)+\tfrac{1}{2}K_+(a_j,\beta_j)\,\hat{\eta}^2+\tfrac{1}{6}\lambda_{\zeta\zeta\zeta}(\alpha_j,\beta_j)\,\zeta^3. \qquad (35)$$

Here,

$$\lambda_{\zeta\zeta\zeta} = -\phi_{\alpha\alpha\alpha}\sin^3\psi+3\phi_{\alpha\alpha\beta}\sin^2\psi\cos\psi-3\phi_{\alpha\beta\beta}\cos^2\psi\sin\psi+\phi_{\beta\beta\beta}\cos^3\psi; \qquad (36)$$

and the third-order derivatives $\phi_{\alpha\alpha\alpha}$, $\phi_{\alpha\alpha\beta}$, $\phi_{\alpha\beta\beta}$ and $\phi_{\beta\beta\beta}$ are

$$\left.\begin{aligned}
\phi_{\alpha\alpha\alpha} &= \frac{|z|}{R}\frac{3}{\alpha|\alpha|}\left\{-\frac{\xi y^2/z^2(1-\xi^2r^2/z^2)^{\frac{3}{2}}}{(1-\xi^2)^2}-\frac{y^2/z^2(1-\xi^2r^2/z^2)^{\frac{1}{2}}}{(1-\xi^2)^2}\alpha\left(\frac{U}{N}\right)\right.\\
&\qquad\left.+\frac{(1-\xi^2r^2/z^2)^{\frac{1}{2}}(2-3\xi^2)}{\xi^3(1-\xi^2)^2}\left(\alpha\frac{U}{N}\right)^2+\frac{[-2\xi^2(1-\xi^2)+(2-3\xi^2)(2\xi^2-1)]}{\xi^4(1-\xi^2)^2(1-\xi^2r^2/z^2)^{\frac{1}{2}}}\left(\alpha\frac{U}{N}\right)^3\right\},\\
\phi_{\alpha\alpha\beta} &= \frac{|z|}{R}\frac{1}{\alpha^2}\frac{\xi y/|z|}{(1-\xi^2)^2\zeta}\left\{(1-\xi^2r^2/z^2)[2(1-\xi^2r^2/z^2)-\xi^2y^2/z^2]\right.\\
&\qquad\left.+2\left(\frac{U}{N}\alpha\right)\frac{(1-\xi^2r^2/z^2)}{\xi}+\frac{(\alpha U/N)(2-3\xi^2)}{\xi^2}\right\},\\
\phi_{\alpha\beta\beta} &= \frac{|z|}{R}\frac{1}{\alpha|\alpha|}\frac{(1-\xi^2r^2/z^2)^{\frac{3}{2}}}{(1-\xi^2)^2\xi}\left\{[2\xi^2y^2/z^2-(1-\xi^2r^2/z^2)]-\frac{\alpha U/N}{\xi}\right\},\\
\phi_{\beta\beta\beta} &= \frac{|z|}{R}\frac{3}{\alpha^2}\frac{y/|z|(1-\xi^2r^2/z^2)^2}{(1-\xi^2)^2}.
\end{aligned}\right\} \qquad (37)$$

Therefore, in preferred directions, the integral for $G^{(c)}$ becomes approximately

$$G^{(c)} \simeq \frac{\exp(i\omega t)}{2(2\pi)^2} \sum_j \frac{1}{R^{\frac{5}{6}}} \left(\frac{2\pi}{|K_+|}\right)^{\frac{1}{2}} \left(\frac{6}{|\lambda_{\xi\xi\xi}|}\right)^{\frac{1}{3}} \frac{\Gamma(\frac{1}{3})}{\sqrt{3}} \exp\left[\tfrac{1}{4}i\pi \operatorname{sgn}(K_+)\right] f(\alpha_j, \beta_j) \times \exp\left[iR\phi(\alpha_j, \beta_j)\right]; \quad (38)$$

j is summed over all points of stationary phase.

5. Results

The asymptotic analysis performed in §4 is formal, and does not provide much insight into the internal-wave system excited by a translating and heaving body. This section discusses the character of the waves excited, the variation in the nature of these waves with changing parameters (velocity of the body U, Brunt–Väisälä frequency N and heave frequency ω), and finally the magnitude of the expressions obtained from the asymptotic analysis performed above.

Lighthill (1959) gave a prescription for calculating surfaces of constant phase for waves in an anisotropic dispersive medium. For a three-dimensional wave system produced by a translating and oscillating body, the surfaces of constant phase are given parametrically by the vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{|A|}{|\alpha|\, U/N(\alpha U/N + \omega/N)(\alpha^2+\beta^2+\gamma^2)} \begin{pmatrix} \alpha[(\alpha U/N + \omega/N)^2 - 1] + U/N(\alpha U/N + \omega N)(\alpha^2+\beta^2+\gamma^2) \\ \beta[(\alpha U/N+\omega/N)^2 - 1] \\ \gamma(\alpha U/N + \omega/N)^2 \end{pmatrix}. \quad (39)$$

From the radiation condition, γ can be eliminated through

$$\gamma = \operatorname{sgn}(z) \frac{[1-(\alpha U/N + \omega/N)^2]^{\frac{1}{2}}}{(\alpha U/N + \omega/N)} (\alpha^2+\beta^2)^{\frac{1}{2}}.$$

α and β are parameters that describe the surface; and A is a constant which defines the particular constant-phase surface. The contours in figures 2–8 were computed by choosing a value of ω/N and taking $|A| = U/N = 1$. Values of α were taken so that $-1 \leqslant \alpha U/N + \omega/N \leqslant 1$. A value of z was selected; and the equation for z was used to determine corresponding values of β. Then x and y were calculated.

There were two aspects of these surfaces to be discussed here. (i) The surfaces display different topological forms, depending upon the value of ω/N and whether U is zero or not. (ii) Most of the surfaces display cusps; and cusp points define preferred directions.

For the purpose of classification, there are five different topological forms which the surfaces of constant phase can take for a translating and/or oscillating excitation source, depending upon the values of ω/N and U. If $U = 0$, the body (excitation source) is at rest, but oscillating, and the surfaces of constant phase are vertical cones. This case is analysed in the appendix.

The other extreme is when $\omega = 0$ and $U \neq 0$. In this case, the surfaces of constant phase are as shown in figure 2. The curve $y = 0$ represents the intersection of the $y = 0$ plane with the constant-phase surface, and is plotted above the x axis. Below the x axis various curves are plotted, representing curves of

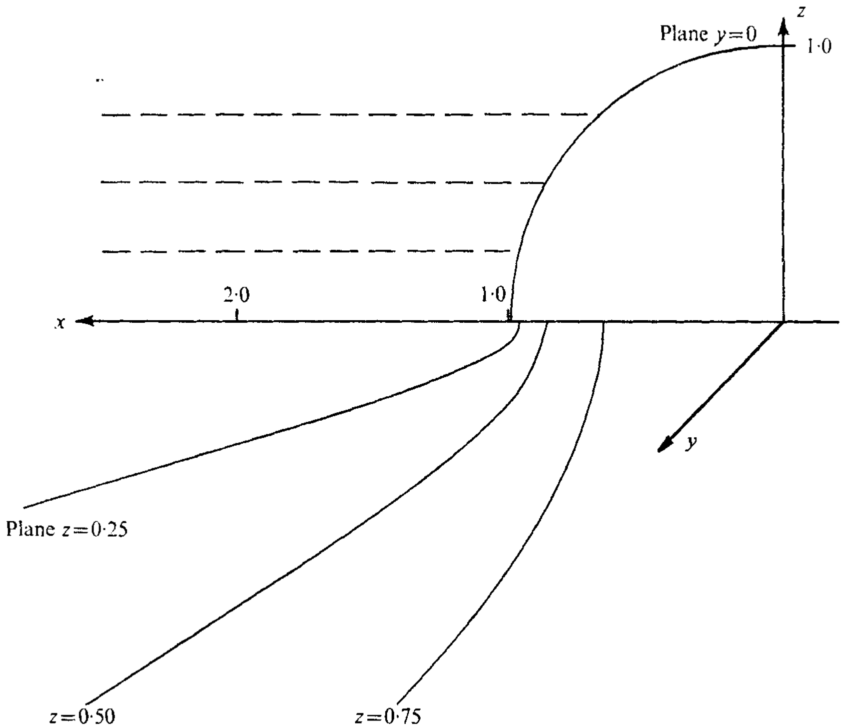

FIGURE 2. Intersection curves through the constant-phase surface for $\omega/N = 0$.

intersection of the constant-phase surface with z = constant planes. The body is at the origin translating with velocity U in the negative x direction, and the wavelength of the waves is $2\pi U/N$. Hence the distance (or wavelength) between two successive crests (surfaces), geometrically similar to the surface displayed in figure 2, is $2\pi U/N$. The wave system is all confined to the region behind the body.

When neither the velocity U nor the frequency ω of oscillation of the body is zero, there are three remaining cases, $\omega < N$, $\omega > N$ and $\omega \approx N$. For the first case, figures 3–5 show three plots of surfaces of constant phase for different values of the ratio ω/N, $\omega/N = 0{\cdot}5$, $0{\cdot}75$ and $0{\cdot}9$. As in figure 2, the curves plotted above the axis are the intersection of the $y = 0$ plane with the constant-phase surfaces, and those curves below the x axis are the intersections for $y > 0$ of the constant-phase surface with various planes parallel to the x, y plane (z = constant planes).

From these plots several features of the surfaces of constant phase for $\omega/N < 1$ can be noted. (i) The waves spread both ahead of (to $x < 0$) and behind ($x > 0$) the body. This feature of the surfaces should be contrasted with the surfaces generated by pure translation (shown in figure 2), where the waves extend only behind the body, as already noted. (ii) The surface of constant phase has two sheets, one confined to the region behind the body ($x > 0$) and one that may extend both ahead and behind it. (iii) The sheet of the surface entirely behind the body exhibits cusps on the curves of intersection with z = constant planes. The cusp feature is very important, and will be discussed later.

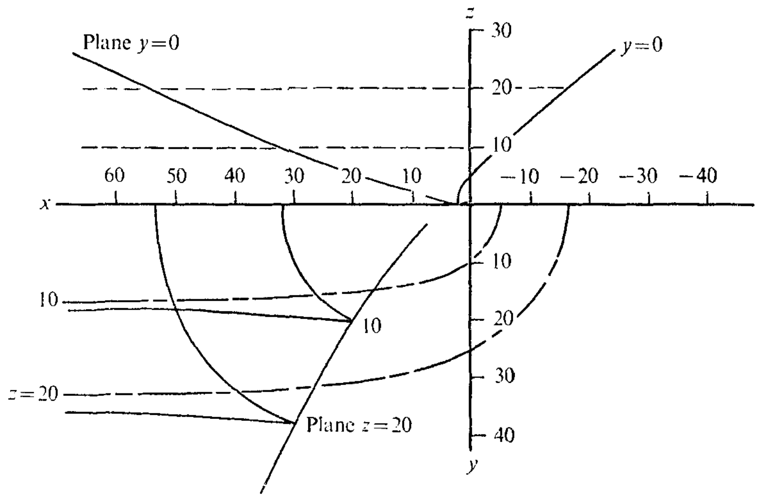

FIGURE 3. Intersection curves through the constant-phase surface for $\omega/N = 0{\cdot}50$.

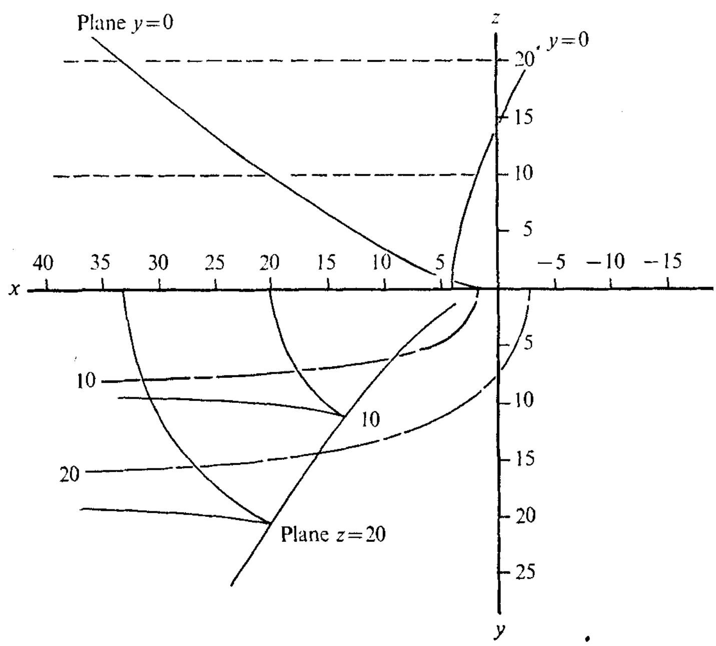

FIGURE 4. Intersection curves through the constant-phase surface for $\omega/N = 0{\cdot}75$.

Finally, comparison of the surfaces for the three values of ω/N, 0·5, 0·75 and 0·90, shows that, as the ratio of ω/N increases towards unity, the general shape of the surfaces remains the same, but the surfaces are contracted in the lateral, y direction while they are extended in the vertical, z direction. Hence they become more upright and more slender in the direction transverse to the motion of the body as $\omega/N = 1$, discussed below.

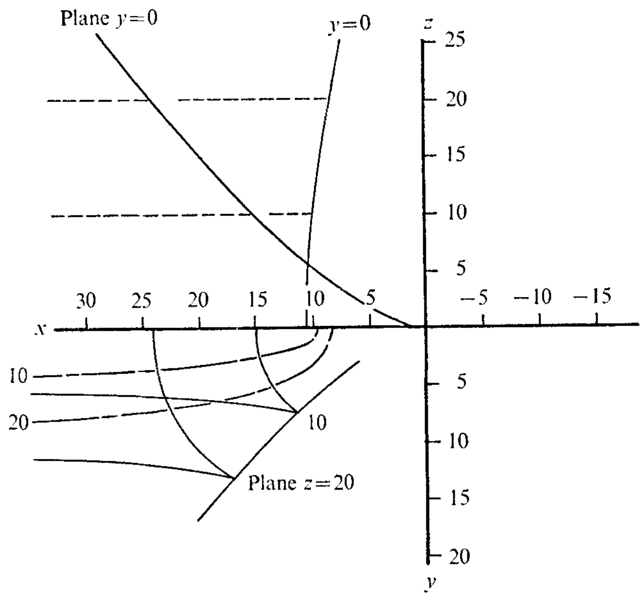

FIGURE 5. Intersection curves through the constant-phase surface for $\omega/N = 0{\cdot}90$.

For $\omega/N > 1$, figures 6–8 show plots of surfaces of constant phase for the values of the ratio $\omega/N = 1{\cdot}1$, $1{\cdot}212$ and $4{\cdot}0$. They are plotted in the same fashion as figures 3–5, z above the x axis and y below. Several points should be mentioned concerning these surfaces. (i) They extend only over a finite region behind the body. This is in contrast with previously shown wave surfaces for $\omega/N < 1$, which extend to infinity in some directions. (ii) As when $\omega/N < 1$, these surfaces display a cusped behaviour at certain points. (iii) They are all topologically alike; and, as when $\omega/N < 1$, they become more elongated vertically and more contracted in the lateral, y direction, as ω/N approaches unity.

It may be somewhat surprising that there are internal waves excited at all when $\omega/N > 1$. When the body is at rest ($U = 0$) and oscillating, internal waves are excited only when the oscillation frequency is less than the Brunt–Väisälä frequency ($\omega/N < 1$). Lighthill (1967) observed that a Doppler effect occurs when the forcing function exciting the waves is both translating and oscillating; and this permits the excitation of internal waves for $\omega/N > 1$.

Consider again figure 3 for the constant-phase surface when $\omega/N = 0{\cdot}5$. The surface behind the body has a cusped behaviour. For example, the curve describing the intersection of the plane $z = 10$ with the constant-phase surface has a cusp point, or a point at which the shape along the curve is discontinuous, at approximately $x = 20, y = 20$ and $z = 10$. The locus of cusp points, or the curve joining adjacent cusps at each vertical level, defines a curve for which x, y and z increase from values near zero out to infinity. A preferred direction, in the sense described in the asymptotic analysis of §4, is one defined by a line passing from the origin through one of the points on the locus of cusps (Lighthill 1959).

The general far-field asymptotic decay of the wave field, with distance from

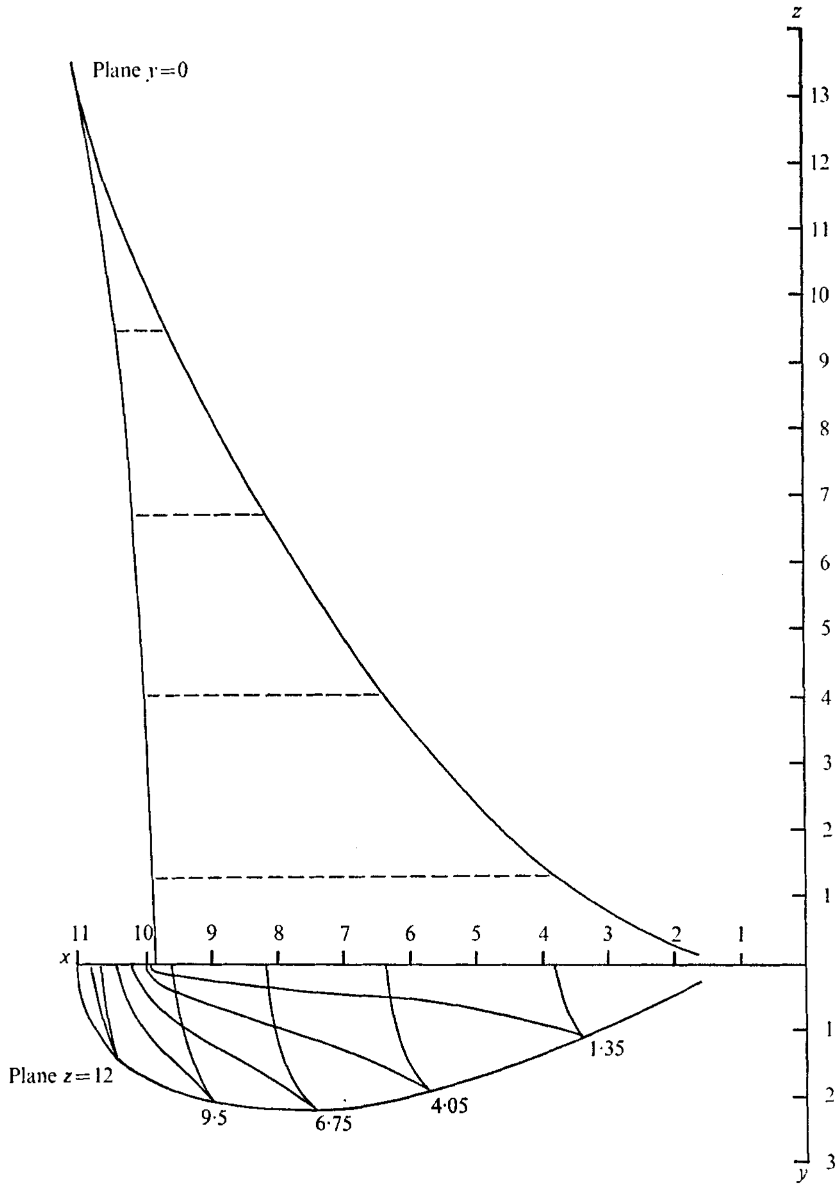

FIGURE 6. Intersection curves through the constant-phase surface for $\omega/N = 1{\cdot}10$.

the excitation source, can be summarized as follows. If a ray drawn in space from the excitation source in any arbitrary direction fails to intersect a constant-phase surface, no waves are effectively excited in the far field. If the ray does intersect a constant-phase surface, the analysis of §4 indicates that the wave amplitude will fall off with increasing distance R from the body as R^{-1}. In certain directions the ray will pass from the origin through a point on the locus of cusps on the constant-phase surface. In one of these directions the wave amplitude falls off as $R^{-\frac{5}{6}}$.

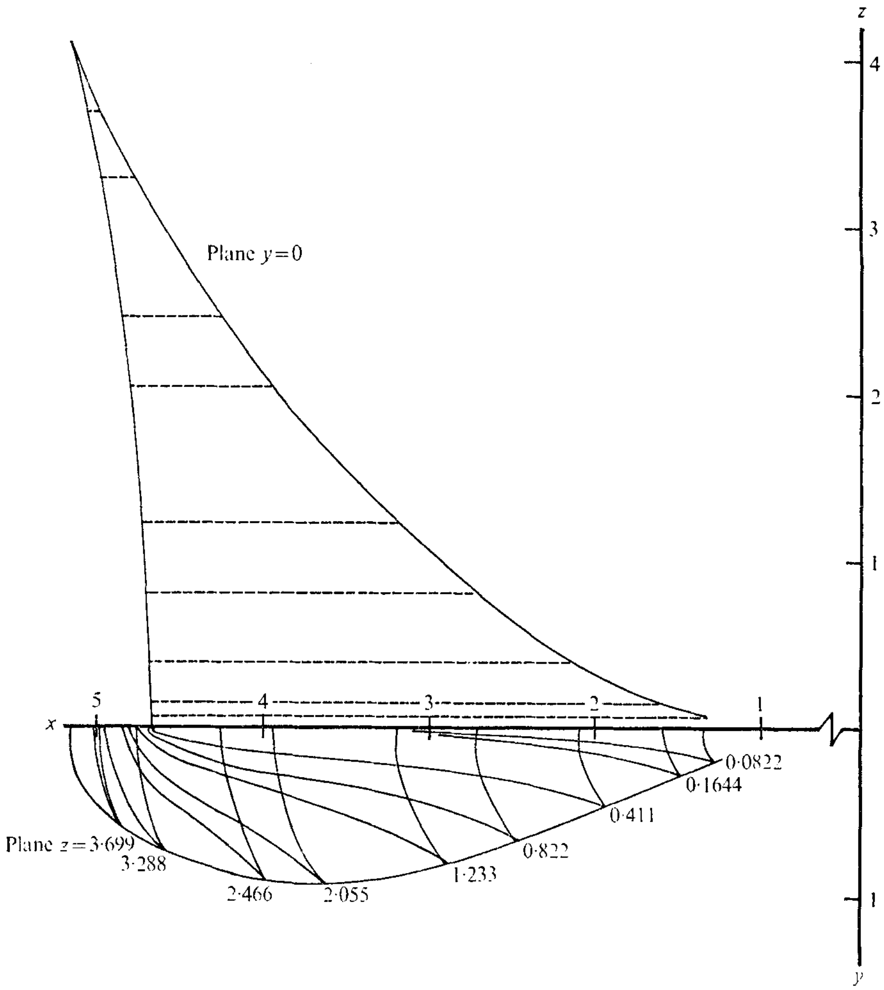

FIGURE 7. Intersection curves through the constant-phase surface for $\omega/N = 1{\cdot}212$.

The expressions for the amplitude of the wave field in preferred directions was calculated for several values of the ratio ω/N. Figure 9 shows the result of these calculations. Here the amplitude function is plotted as a function of the angle θ that the preferred direction makes with the horizontal. Several features of this plot should be noted. It appears that the angles of the preferred directions above horizontal do not exceed 60°, and in particular, that the preferred directions never become vertical. Also, the maximum angle above horizontal of the preferred directions decreases rapidly when ω/N becomes larger than unity. Therefore, even though the amplitude function increases fairly rapidly in magnitude with increasing ω/N, the angle with the horizontal over which the waves spread is small.

Finally, for $\omega/N = 0{\cdot}9$, the curve of the amplitude function shows a sharp increase around $\theta = 55°$. This sharp increase arises because the value of $\lambda_{\zeta\zeta\zeta}$ in the denominator of the Green's function (38) vanishes. Several other calcula-

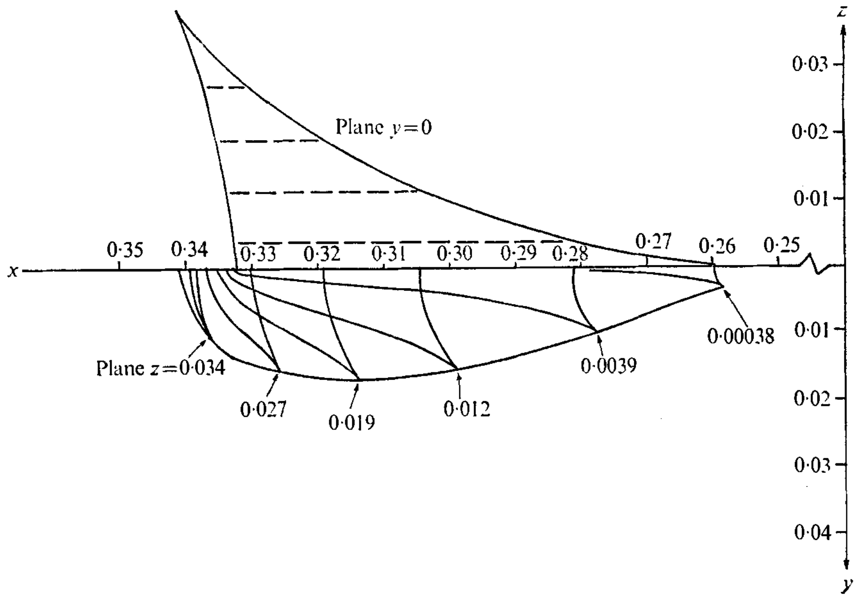

FIGURE 8. Intersection curves through the constant-phase surface for $\omega/N = 4{\cdot}00$.

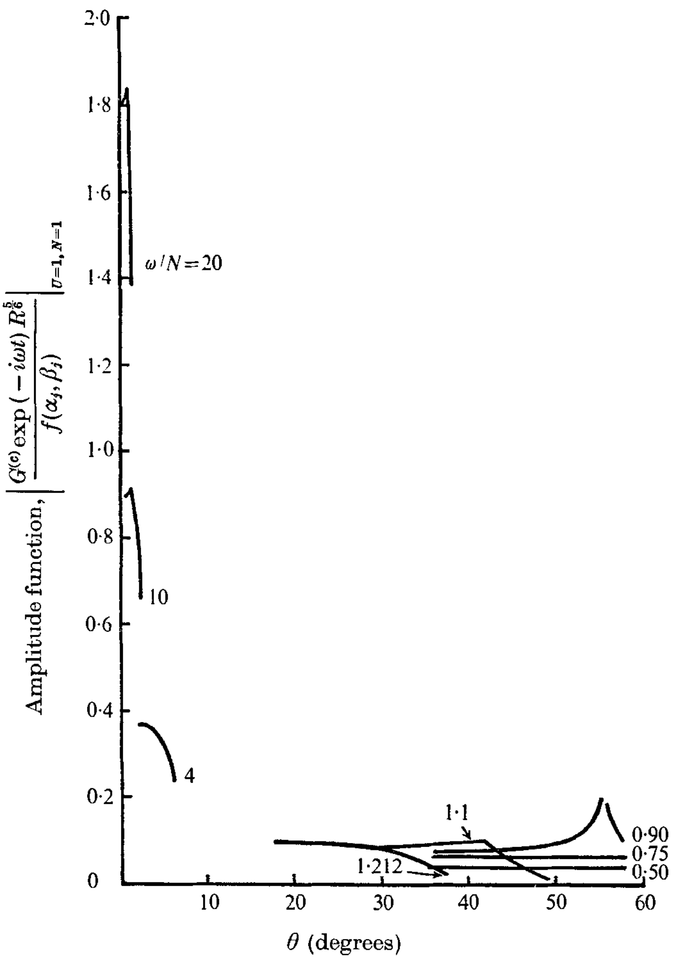

FIGURE 9. Amplitude function along preferred directions as a function of angle above horizontal for various values of the ratio of heave frequency ω to Brunt–Väisälä frequency N.

tions were made for ω/N in the range $0{\cdot}8 < \omega/N < 1{\cdot}0$ and similar behaviour was found. This behaviour is attributed to failure of the numerical procedure to accurately calculate $\lambda_{\zeta\zeta\zeta}$ when $\omega/N \rightarrow 1$. The asymptotic procedure outlined in §4 breaks down when $\omega = N$, and apparently the accuracy required to calculate $\lambda_{\zeta\zeta\zeta}$ for the asymptotic expressions is greater. Detailed analysis of the case when $\omega = N$ has not yet been carried out.

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Appendix. Internal waves excited by an oscillating body at rest

In this appendix we calculate the internal-wave system excited by an axially symmetric (about the vertical axis) slender body oscillating with a frequency below the Brunt–Väisälä frequency N (see figure 10). The body is assumed to be slender, and to execute small-amplitude oscillations, in the sense that both the vertical dimension or thickness of the body, compared with its horizontal dimension, is small, and the amplitude of the oscillation compared with its thickness is small. The fluid is assumed to be exponentially stratified, so that N is constant, and the Boussinesq approximation is made. The fundamental linear equation governing the vertical displacement ζ is (3).

Internal waves from an oscillating source or vertically oriented dipole

For the cases of interest the source functions Q are given by

$$Q_1 = \exp(i\omega t)\int_\xi\int_\eta d\xi d\eta \delta(x-\xi)\,\delta(y-\eta)\,\delta(z)\,\sigma(\xi,\eta), \tag{A 1}$$

$$Q_2 = \exp(i\omega t)\int_\xi\int_\eta d\xi d\eta\int_{-\epsilon}^{\epsilon} d\zeta\,\delta(x-\xi)\,\delta(y-\eta)\,\delta_\zeta(z-\zeta)\,\hat{d}(\xi,\eta). \tag{A2}$$

Q_1 represents a distribution of sources located along the $z = 0$ plane and oscillating in magnitude with frequency ω; Q_2 represents a distribution of dipoles with axes directed upward, oscillating in magnitude with frequency ω and located along the $z = 0$ plane. Any oscillating slender body for which the plane of the body is horizontal and at $z = 0$ can be represented by a superposition of these sources and dipoles.

Fourier-transform techniques are used to obtain the Green's function solutions $G^{(s)}$, $G^{(d)}$ for the oscillating unit source and dipole, respectively. For convenience, the source and dipole are located at the origin. Hence, $G^{(s)}$ and $G^{(d)}$ satisfy

$$[\partial_z^2\partial_t^2 + (\partial_t^2+N^2)(\partial_x^2+\partial_y^2)]\begin{Bmatrix}G^{(s)}\\ G^{(d)}\end{Bmatrix} = i\omega\exp(i\omega t)\,\delta(x)\,\delta(y)\begin{Bmatrix}\partial_z\delta(z)\\ -\partial_z^2\delta(z)\end{Bmatrix}. \tag{A 3}$$

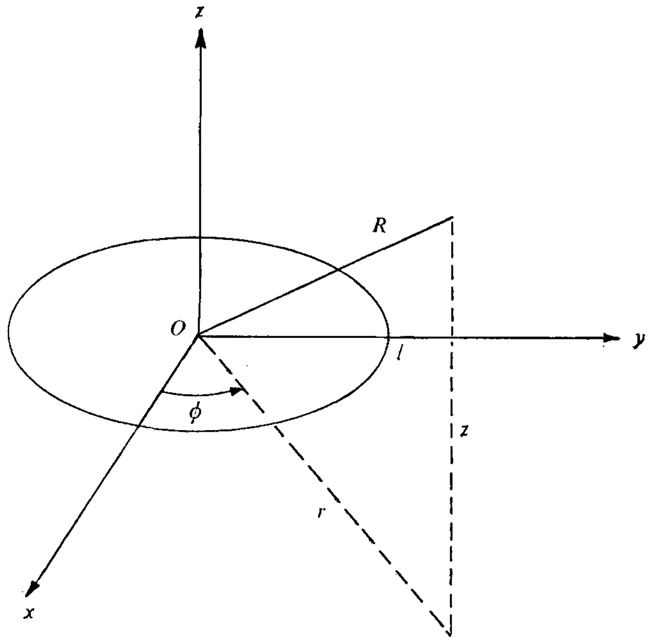

FIGURE 10. Reference axes used for the analysis of the axisymmetric internal-wave system excited by an oscillating body.

Fourier transforms, as defined in (10), are used to solve (A 3). As in the text, integration over the wavenumber γ can be carried out explicitly by using residue theory, and by applying the Sommerfeld radiation condition. Then the solution for the Green's function can be written as

$$\begin{Bmatrix} G^{(s)} \\ G^{(d)} \end{Bmatrix} = \frac{-i\,\mathrm{sgn}\,(z)}{2\omega(2\pi)^2}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \begin{Bmatrix} 1 \\ -i\,\mathrm{sgn}\, z\dfrac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega}(\alpha^2+\beta^2)^{\frac{1}{2}} \end{Bmatrix}$$
$$\times \exp\{i(\alpha x+\beta y+\frac{N^2-\omega^2}{\omega}(\alpha^2+\beta^2)^{\frac{1}{2}}|z|+\omega t)\}. \quad \text{(A 4)}$$

The integrations over the wavenumbers can be carried out explicitly, using results of Olver (1964) and Erdélyi *et al.* (1954, vol. 2), giving

$$G^{(s)} = \frac{-\,\mathrm{sgn}\, z \exp{(i\omega t)}}{2\omega\,(2\pi)}\,\frac{\omega^2(N^2-\omega^2)^{\frac{1}{2}}\,|z|}{[\omega^2 r^2-(N^2-\omega^2)\,z^2]^{\frac{3}{2}}}, \quad \text{(A 5)}$$

and
$$G^{(d)} = \frac{\exp{(i\omega t)}}{2\omega\,(2\pi)}\,\omega^2\,(N^2-\omega^2)^{\frac{1}{2}}\,\frac{\omega^2 r^2+2(N^2-\omega^2)\,z^2}{[\omega^2 r^2-(N^2-\omega^2)\,z^2)]^{\frac{5}{2}}}. \quad \text{(A 6)}$$

Both $G^{(s)}$ and $G^{(d)}$ are singular along the conical surfaces determined by

$$z = \pm\,\frac{\omega}{(N^2-\omega^2)^{\frac{1}{2}}}\,r. \quad \text{(A 7)}$$

In these preferred directions $G^{(s)}$ and $G^{(d)}$ become infinite. However, in other regions of the fluid the internal-wave amplitudes are finite. If $R = [r^2+z^2]^{\frac{1}{2}}$ is the distance from the origin to the point of observation, the internal-wave amplitude excited by an oscillating point source decays with distance as $G^{(s)} \propto 1/R^2$, while that excited by a dipole falls off as $G^{(d)} \propto 1/R^3$.

Internal waves from an oscillating axisymmetric body

Now consider a general axisymmetric distribution σ of oscillating sources or dipoles (σ is only a function of r). Denote by $G(x,y,z,t)$ either $G^{(s)}$ or $G^{(d)}$. The general internal-wave field excited by this distribution can then be written as

$$\zeta = \int_0^{2\pi} d\psi \int_0^l r'\,dr'\sigma(r')\,G(x-\xi, y-\eta, z, t), \tag{A 8}$$

where $\quad r' = (\xi^2+\eta^2)^{\frac{1}{2}} \quad \text{and} \quad \psi = \arctan(\eta/\xi).$

Substitution of (A 4) into (A 8) yields the following expressions for the vertical displacement produced by a distribution of sources and of dipoles, respectively:

$$\zeta^{(s)} = \frac{-i\,\mathrm{sgn}\,(z)}{2\omega}\exp(i\omega t)\int_0^\infty \rho\,d\rho\exp\left[i\rho\frac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega}|z|\right]J_0(\rho r)\int_0^l r'\,dr'\sigma(r')\,J_0(\rho r'), \tag{A 9}$$

$$\zeta^{(d)} = \frac{-\exp(i\omega t)}{2\omega}\frac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega}\int_0^\infty \rho^2\,d\rho\exp\left[i\rho\frac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega}|z|\right]J_0(\rho r) \times\int_0^l r'\,dr'\,\hat{d}(r')\,J_0(\rho r'). \tag{A 10}$$

J_0 is the zero-order Bessel function, $\rho = (\alpha^2+\beta^2)^{\frac{1}{2}}$; and results from Magnus & Oberhettinger (1949) have been used.

To apply the boundary conditions, these expressions are written in terms of the vertical velocity $w = \partial\zeta/\partial t$. Each of the distribution functions $\sigma(r)$ and $\hat{d}(r)$ can be determined by application of boundary conditions at the body surface and at the remaining portion of the x, y plane. For a distribution of sources, the velocity above and below the region $z = 0$, $0 \leqslant r \leqslant l$ is directed away from this region. By symmetry considerations, the velocity at $z = 0$ and $l < r < \infty$ will be zero. Thus,

$$\left.\begin{aligned} &w^{(s)}(r, z = \pm 0, t) = \pm W(r)\exp(i\omega t), \quad 0 \leqslant r \leqslant l,\\ &w^{(s)}(r, z = 0, t) = 0, \quad l < r < \infty, \end{aligned}\right\} \tag{A11}$$

$$w^{(s)}(r, z = \pm 0, t) = \pm\tfrac{1}{2}\exp(i\omega t)\int_0^\infty \rho\,d\rho\,J_0(\rho r)\int_0^l r'\,dr'\sigma(r')\,J_0(\rho r'). \tag{A12}$$

This integral equation for the determination of $\sigma(r)$ can be inverted by using the Hankel transform relation (see e.g. Erdélyi, vol. 2, p. 5). Then

$$\sigma(r) = 2W(r); \tag{A 13}$$

and the source distribution is determined in terms of the velocity distribution along the surface of the slender body.

Likewise the dipole distribution $\hat{d}$ can be used to satisfy the boundary conditions when the velocity at the top and bottom of the body have the same magnitude and direction (heave motion of the body). In this case, the problem has mixed boundary conditions along the x, y plane. The velocity given on $z = 0$, $0 \leqslant r \leqslant l$, is

$$w^{(d)}(r, z = 0, t) = \hat{W}(r) \exp(i\omega t); \tag{A 14}$$

and the integral of the velocity from $z = 0-$ to $z = 0+$ must be zero from $l < r < \infty$:

$$-\int_{z=0-}^{z=0+} w^{(d)}(r, z', t)\, dz' = 0. \tag{A15}$$

Equations (A 14) and (A 15) become

$$\left.\begin{aligned} \hat{W}(r) &= \frac{-i}{2} \frac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega} \int_0^\infty \rho^2 d\rho\, J_0(\rho r) D(\rho) \quad \text{for} \quad 0 \leqslant r \leqslant l, \\ 0 &= \int_0^\infty \rho\, d\rho\, J_0(\rho r) D(\rho) \quad \text{for} \quad r > l, \end{aligned}\right\} \tag{A 16}$$

where

$$D(\rho) = \int_0^l r' dr' \hat{d}(r') J_0(\rho r').$$

The dipole distribution $\hat{d}(r)$ can be expanded in terms of Legendre polynomials P_n as

$$\left.\begin{aligned} \hat{d}(r) &= [1-(r/l)^2] \sum_{n=0}^{\infty} B_n P_n[1-(r/l)^2 2] \quad \text{for} \quad 0 \leqslant r/l < 1, \\ &= 0 \quad \text{for} \quad 1 < r/l < \infty, \end{aligned}\right\} \tag{A 17}$$

yielding the velocity along $z = 0$, $0 \leqslant r \leqslant l$,

$$\hat{W}(r) = \frac{-i}{2} \frac{(N^2-\omega^2)^{\frac{1}{2}}}{\omega} \sum_{n=0}^{\infty} \frac{2B_n}{2n+1} \{n^2 F(n+\tfrac{1}{2}, \tfrac{1}{2}-n; 1; r^2/l^2) + (n+1)^2 F(n+\tfrac{3}{2}, -\tfrac{1}{2}-n; 1; r^2/l^2)\}. \tag{A 18}$$

The coefficients B_n can be determined in principle, for a given velocity distribution $\hat{W}(r)$. Each component function $[1-(r/l)^2] P_n[1-2(r/l)^2]$ of the expansion for the dipole distribution $\hat{d}(r)$ gives rise to an internal-wave field, which will be calculated asymptotically in the far field below. The source function $\sigma(r)$ is also expanded into component functions for which the asymptotic, far-field waves are calculated below.

Amplitude of internal waves far from an oscillating axisymmetric body

Return to the expressions for $\zeta^{(s)}$ and $\zeta^{(d)}$ given by (A 9) and (A 10), assuming that $\sigma(r)$ and $\hat{d}(r)$ are chosen to satisfy proper boundary conditions. For the source and dipole distributions take the component functions

$$\left.\begin{aligned} \sigma(r) &= P_n(1-2(r/l)^2) \quad \text{for} \quad 0 \leqslant r/l \leqslant 1, \\ &= 0 \quad \text{for} \quad 1 < r/l < \infty, \end{aligned}\right\} \tag{A 19}$$

$$\left.\begin{aligned} \hat{d}(r) &= [1-(r/l)^2] P_n(1-2(r/l)^2) \quad \text{for} \quad 0 \leqslant r/l \leqslant 1, \\ &= 0 \quad \text{for} \quad 1 < r/l < \infty. \end{aligned}\right\} \tag{A 20}$$

Asymptotic evaluation of (A 9) and (A 10) for large distances from the body, $R/l = (r^2+z^2)^{\frac{1}{2}}/l \gg 1$, can be performed. With some manipulation, it can be shown that outside the regions around the preferred directions

$$\zeta^{(s)} \sim (l/R)^{2n+2}, \quad \zeta^{(d)} \sim (l/R)^{2n+1} \tag{A21}$$

to lowest order in l/R. In the preferred directions, each component function varies with distance as

$$\zeta^{(s)} \propto (l/R)^{\frac{1}{2}}, \quad \zeta^{(d)} \propto (l/R)^{\frac{1}{2}} \tag{A 22}$$

to lowest order in l/R. Therefore, the amplitude $\zeta^{(0)}$ of the internal waves excited by an axisymmetric body of horizontal dimension l can be written as

$$\zeta^{(0)} = C(u^{(t)}/\omega)\,(l/R)^{\frac{1}{2}}. \tag{A 23}$$

C is a constant, which depends upon the particular type of excitation mechanism (bobbing body, fluid breathing body, etc.), and which is of order unity. $u^{(t)}$ is the magnitude of the velocity of oscillation at the body surface. l is the radius of the body. R is radial distance from the body. The quantity $u^{(t)}/\omega$ is the magnitude of the fluid displacement at the surface of the body. Since the problem has been linearized, $\zeta^{(0)}$ must be proportional to this displacement.

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